

Kink scaling functions in 2D non-integrable quantum field theories

G. Mussardo^{1,2}, V. Riva³, G. Sotkov^{4,*} and G. Delfino^{1,2}

¹*International School for Advanced Studies
Via Beirut 1, 34100 Trieste, Italy*

²*Istituto Nazionale di Fisica Nucleare, Sezione di Trieste*

³*Rudolf Peierls Centre for Theoretical Physics, University of Oxford
1 Keble Road, Oxford, OX1 3NP, UK
and Wolfson College, Oxford*

⁴*Departamento de Fisica, Universidade Federal do Espirito Santo
29060-900 Vitoria, Espirito Santo, Brazil*

Abstract

We determine the semiclassical energy levels for the ϕ^4 field theory in the broken symmetry phase on a 2D cylindrical geometry with antiperiodic boundary conditions by quantizing the appropriate finite-volume kink solutions. The analytic form of the kink scaling functions for arbitrary size of the system allows us to describe the flow between the twisted sector of $c = 1$ CFT in the UV region and the massive particles in the IR limit. Kink-creating operators are shown to correspond in the UV limit to disorder fields of the $c = 1$ CFT. The problem of the finite-volume spectrum for generic 2D Landau-Ginzburg models is also discussed.

E-mail addresses: mussardo@sissa.it, v.riva1@physics.ox.ac.uk, sotkov@inrne.bas.bg
delfino@sissa.it

* On leave of absence - Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Tsarigradsko Chaussee 72, BG-1784, Sofia, Bulgaria.

1 Introduction

The universal thermodynamical properties of statistical systems with multicritical behavior are described, in mean-field approximation, by appropriate Landau–Ginzburg (LG) field theories:

$$V_l(\phi) = \sum_{k=1}^l \lambda_k \phi^{2k-2} \quad , \quad l = 3, 4, \dots \quad (1.1)$$

Structural (commensurate–incommensurate) phase transitions [1], interface phenomena in ordered and disordered media [2] and phase structure of ferromagnetic systems (see for instance [3]) provide few examples for the applications of the simplest ϕ^4 and ϕ^6 LG models to statistical mechanics and condensed matter physics. In two dimensions, the LG potentials (1.1) appear also in the description of the relevant perturbations of Virasoro minimal models of conformal field theory [4], as well as of the renormalization group flows between them.

The physical quantities associated with a field theory — partition function, energy spectrum, correlation functions, etc. — strongly depend on the geometry of the considered problem (cylindrical, strip, plane, etc.), on the boundary conditions chosen (periodic, Dirichlet, etc.) and on the range of the values of the couplings λ_k . For several integrable quantum field theories in 2D, the above quantities have been exactly computed in finite volume with the so-called Thermodynamics Bethe Ansatz method [5] or Destri–deVega equations [6]. These techniques, however, require the integrability of the model, and cannot be applied to the LG theories (1.1), due to their non-integrable nature. In this case, the analysis of the finite-size effects is based on approximative methods as perturbative renormalization group (see [2, 3] and references therein), transfer integral techniques [1] and numerical methods.

The low temperature (broken symmetry) phase of these models exhibits, however, specific features — multiple degenerate vacua, non-trivial topological sectors and non-perturbative kink solutions (domain walls) — which require certain improvements of the standard perturbative methods. The non-perturbative semiclassical expansion [7] is known to be an effective method for the quantization of the kink solutions in an infinite volume, independently of the integrability of the model. Its recent extension to finite geometries [8, 9] allowed us to derive analytic expressions for the scaling functions of the Sine–Gordon model defined on a cylinder with quasi-periodic b.c. (i.e. in the one-kink sector) and on a strip with Dirichlet b.c.’s. It is then natural to address the problem of the finite-size effects in 2D LG models within the context of the semiclassical quantization of kinks in finite volume.

The present paper is devoted to the derivation of the scaling functions of the 2D ϕ^4 theory on a cylindrical geometry with *antiperiodic* b.c. $\phi(x+R) = -\phi(x)$, which for this model corresponds to consider a single kink on the cylinder. This continues our analysis of finite-size effects in the ϕ^4 model, which begun in [10] with the derivation of the finite-volume form factors and spectral functions for the same kind of geometry.

From the mathematical point of view, the derivation of the scaling functions for the ϕ^4 theory on the *twisted* cylinder is analogous to the one performed in [8] for the Sine–Gordon

model on a cylinder with *quasi-periodic* b.c.. This is due to the fact that the finite volume kinks are expressed in both cases in terms of a Jacobi elliptic function, and the computation of the corresponding energy levels is therefore based on the solution of the so-called Lamé equation. Besides a minor technical difference (the equation appears now in a more complicated form, the so-called $N = 2$ Lamé form), an important new feature emerges in the antiperiodic case: the oscillating background cannot be defined for any value of the size of the system, so that the complete description of the problem is achieved in this case by also including a constant background below a specific value of the size.

Our main result, presented in Sect. 2, consists in the analytic expression of the kink scaling functions (for arbitrary value of the size of the system R), which describes the flow between the twisted sector of $c = 1$ CFT in the UV region and the massive particles in the $Q = \pm 1$ topological sectors of the broken ϕ^4 theory in the infrared (IR) limit. This Section also includes a comparison between the large- R corrections to the kink masses, as obtained from the IR asymptotic behaviour of the scaling functions, and the values expected from the infinite-volume scattering data through Luscher's theory [11].

A detailed study of the UV regime is left to Sect. 3. Here we analyse the properties of the $c = 1$ CFT fields that play the role of creating operators for the ϕ^4 kinks, as well as of the kinks of generic LG models. It turns out that for Z_2 -invariant polynomial potentials (in their broken phase) the disorder field μ of dimension $1/8$ (and its descendants) from the twisted sector of the $c = 1$ CFT are the only operators local with respect to the potential and carrying topological (Z_2) charges. Therefore they must describe the UV limit of the LG-kinks.

Sect. 3 actually begins with the more familiar discussion of soliton-creating operators for the Sine-Gordon model in the winding (i.e. quasiperiodic) sector. Due to the compactification of the field, indeed, this theory admits more types of b.c., including the antiperiodic ones. We have then devoted Sect. 4 to the analysis of this interestingly rich model, which displays two types of non-trivial classical solutions in finite volume, respecting two different b.c.'s (quasiperiodic and antiperiodic). Their UV limits are described, respectively, by the standard soliton-creating operators from the winding sector of $c = 1$ CFT and by the disorder field in its twisted sector, i.e. that one which creates the Z_2 charged kinks. The two corresponding types of scaling functions are given explicitly, and their difference is observed at any finite volume, except for their identical IR limits. It is therefore clear that passing from periodic to Z_2 -symmetric polynomial potentials only the kink-type (antiperiodic) solution survives, which explain why the finite volume kink-type solutions of SG and ϕ^4 models (as well as their UV limits) share many common properties.

The explicit analytic form obtained in the present paper for the scaling functions of the ϕ^4 model (and in previous works [8, 9] for the Sine-Gordon model) is intrinsically related to the fact that the stability equations to be solved are of Lamé type, and the corresponding solutions are well known. As we shall show in Sect. 5, similar construction for ϕ^6 and higher ($l \geq 5$) LG models leads again to Schrödinger-like equations for periodic potentials, but it turns out that these are more complicated generalizations of the Lamé equation. The derivation of the finite-volume energy spectrum of these models thus depends on the further progress that will

be achieved in the future on their analytical or numerical solutions.

2 Semiclassical quantization of the broken ϕ^4 theory in finite volume

The standard perturbative methods of QFT's in D -dimensions (including the $D = 2$ case we are interested in) are known to be inefficient for the description of the quantum effects in the topologically non-trivial sectors of an important class of theories with non-linear interactions and multiple degenerate vacua. As a rule, such theories admit finite-energy non-perturbative classical solutions (kinks, vortices, monopoles etc.) carrying topological charges. The quantization of these solutions (both static and time-dependent) requires non-perturbative techniques, developed by Dashen, Hasslacher and Neveu (DHN) in [7] for theories in infinite volume. The DHN method consists, for static backgrounds, in splitting the field $\phi(x, t)$ in terms of the classical solution and its quantum fluctuations, i.e.

$$\phi(x, t) = \phi_{cl}(x) + \eta(x, t) \quad , \quad \eta(x, t) = \sum_k e^{i\omega_k t} \eta_k(x) \quad ,$$

and in further expanding the Lagrangian of the theory in powers of η , keeping only the quadratic terms. As a result of this procedure, $\eta_k(x)$ satisfies the so called “stability equation”

$$\left[-\frac{d^2}{dx^2} + V''(\phi_{cl}) \right] \eta_k(x) = \omega_k^2 \eta_k(x) \quad , \quad (2.1)$$

together with certain boundary conditions. The semiclassical energy levels in each sector are then built in terms of the energy of the corresponding classical solution and the eigenvalues ω_i of the Schrödinger-like equation (2.1), i.e.

$$E_{\{n_i\}} = \mathcal{E}_{cl} + \hbar \sum_k \left(n_k + \frac{1}{2} \right) \omega_k + O(\hbar^2) \quad , \quad (2.2)$$

where n_k are non-negative integers. In particular the ground state energy in each sector is obtained by choosing all $n_k = 0$ and it is therefore given by¹

$$E_0 = \mathcal{E}_{cl} + \frac{\hbar}{2} \sum_k \omega_k + O(\hbar^2) \quad . \quad (2.3)$$

In our recent papers [8, 9], we have extended this technique to the study of soliton quantization in the Sine-Gordon model on the cylinder (with periodic b.c.) and on a strip with Dirichlet b.c.. This Section is devoted to the quantization of the kinks of the ϕ^4 theory in the \mathbb{Z}_2 broken symmetry phase, defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - V(\phi) \quad , \quad \text{with} \quad V(\phi) = \frac{\lambda}{4} \left(\phi^2 - \frac{m^2}{\lambda} \right)^2 \quad , \quad (2.4)$$

¹From now on we will fix $\hbar = 1$, since the semiclassical expansion in \hbar is equivalent to the expansion in the interaction coupling λ .

on a cylinder with the antiperiodic b.c.'s

$$\phi(x + R) = -\phi(x) , \quad (2.5)$$

imposed. In order to fix the ideas and the notations, we first shortly review the DHN method for the quantization of ϕ^4 -kinks in infinite volume.

2.1 Infinite volume kinks

The static solutions of the equation of motion associated to the potential (2.4) can be obtained by integrating the following first order equation

$$\frac{1}{2} \left(\frac{\partial \bar{\phi}_{cl}}{\partial \bar{x}} \right)^2 = \frac{1}{4} (\bar{\phi}_{cl}^2 - \bar{\phi}_0^2) (\bar{\phi}_{cl}^2 - 2 + \bar{\phi}_0^2) , \quad (2.6)$$

where we have rescaled the variables as

$$\bar{\phi} = \frac{\sqrt{\lambda}}{m} \phi , \quad \bar{x} = mx , \quad (2.7)$$

and ϕ_0 is an arbitrary constant defined by $V(\phi_0) = -A$, i.e.

$$\frac{1}{2} \left(\frac{\partial \phi_{cl}}{\partial x} \right)^2 = V(\phi_{cl}) + A .$$

In infinite volume we have to impose as b.c. that the classical field reaches the minima of the potential at $x \rightarrow \pm\infty$, i.e. $\bar{\phi}_{cl}(\pm\infty) = \pm 1$. This corresponds to choosing the value $\bar{\phi}_0 = 1$ for the arbitrary constant in (2.6), and, as a consequence, we find the well-known kink solution

$$\bar{\phi}_{cl}(x) = \tanh \left(\frac{\bar{x} - \bar{x}_0}{\sqrt{2}} \right) , \quad (2.8)$$

shown in Fig. 1, which has classical energy $\mathcal{E}_{cl} = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda}$.

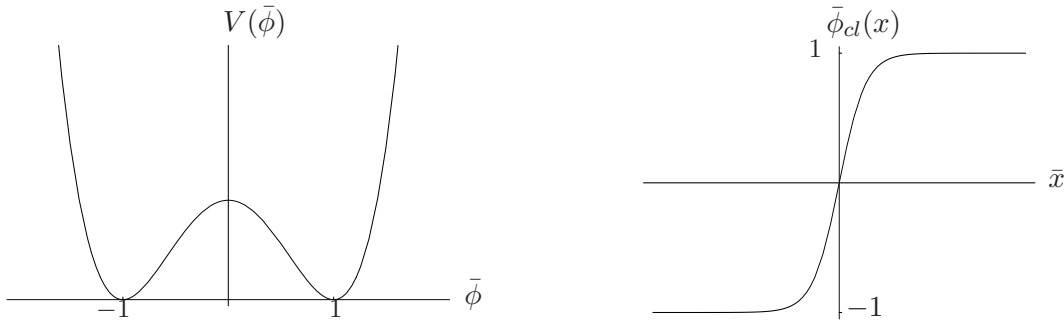


Figure 1: Potential (2.4) and infinite-volume kink (2.8) with $x_0 = 0$.

The stability equation (2.1) around this background can be cast in the hypergeometric form in the variable $z = \frac{1}{2}(1 + \tanh \frac{\bar{x}}{\sqrt{2}})$, and the solution is expressed in terms of the hypergeometric function $F(\alpha, \beta, \gamma; z)$ as

$$\eta(x) = z^{\sqrt{1 - \frac{\omega^2}{2m^2}}} (1 - z)^{-\sqrt{1 - \frac{\omega^2}{2m^2}}} F \left(3, -2, 1 + 2\sqrt{1 - \frac{\omega^2}{2m^2}}; z \right) .$$

The corresponding spectrum is given by the two discrete eigenvalues

$$\omega_0^2 = 0, \quad \text{with} \quad \eta_0(x) = \frac{1}{\cosh^2 \frac{\bar{x}}{\sqrt{2}}}, \quad (2.9)$$

and

$$\omega_1^2 = \frac{3}{2} m^2, \quad \text{with} \quad \eta_1(x) = \frac{\sinh \frac{\bar{x}}{\sqrt{2}}}{\cosh^2 \frac{\bar{x}}{\sqrt{2}}}, \quad (2.10)$$

plus the continuous part, labelled by $q \in \mathbb{R}$,

$$\omega_q^2 = m^2 \left(2 + \frac{1}{2} q^2 \right), \quad \text{with} \quad \eta_q(x) = e^{iq\bar{x}/\sqrt{2}} \left(3 \tanh^2 \frac{\bar{x}}{\sqrt{2}} - 1 - q^2 - 3iq \tanh \frac{\bar{x}}{\sqrt{2}} \right). \quad (2.11)$$

The presence of the zero mode ω_0 is due to the arbitrary position of the center of mass x_0 in (2.8), while ω_1 and ω_q represent, respectively, an internal excitation of the kink particle and the scattering of the kink with mesons² of mass $\sqrt{2}m$ and momentum $mq/\sqrt{2}$.

The semiclassical correction to the kink mass can be now computed as the difference between the ground state energy in the kink sector and the one of the vacuum sector, plus a mass counterterm due to normal ordering:

$$M = \mathcal{E}_{cl} + \frac{1}{2} m \sqrt{\frac{3}{2}} + \frac{1}{2} \sum_n \left[m \sqrt{2 + \frac{1}{2} q_n^2} - \sqrt{k_n^2 + 2m^2} \right] - \frac{1}{2} \delta m^2 \int_{-\infty}^{\infty} dx \left[\phi_{cl}^2(x) - \frac{m^2}{\lambda} \right], \quad (2.12)$$

with

$$\delta m^2 = \frac{3\lambda}{4\pi} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{k^2 + 2m^2}}. \quad (2.13)$$

The discrete values q_n and k_n are obtained by putting the system in a big finite volume of size R with periodic boundary conditions:

$$2n\pi = k_n R = q_n \frac{mR}{\sqrt{2}} + \delta(q_n), \quad (2.14)$$

where the phase shift $\delta(q)$ is extracted from $\eta_q(x)$ in (2.11) as

$$\eta_q(x) \xrightarrow{x \rightarrow \pm\infty} e^{i \left[q \frac{m\bar{x}}{\sqrt{2}} \pm \frac{1}{2} \delta(q) \right]}, \quad \delta(q) = -2 \arctan \left(\frac{3q}{2 - q^2} \right). \quad (2.15)$$

Sending $R \rightarrow \infty$ and computing the integrals one finally has

$$M = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} + m \left(\frac{1}{6} \sqrt{\frac{3}{2}} - \frac{3}{\pi\sqrt{2}} \right). \quad (2.16)$$

Notice that, from the knowledge of this quantity, one can extract a rough estimate of the value of couplings at which the broken ϕ^4 theory actually describes the Ising model. It is well known,

²The mesons represent the excitations over the vacua, i.e. the constant backgrounds $\phi_{\pm} = \pm \frac{m}{\sqrt{\lambda}}$, therefore their square mass is given by $V''(\phi_{\pm}) = 2m^2$.

in fact, that perturbing the conformal gaussian theory $\mathcal{L}_G = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi)$ with the potential (2.4) one can have different renormalization group trajectories depending on the values of the dimensionless coupling λ/m^2 . The universality class of the Ising model is described by the situation in which the infrared point is not a massive theory but rather another conformal field theory, with central charge $c = 1/2$. Therefore, we can estimate semiclassically the corresponding value of λ/m^2 by imposing the vanishing of the mass (2.16), which gives $\lambda/m^2 \simeq 2$. The large value of this quantity suggests, however, that the one-loop order in the semiclassical expansion in λ/m^2 can hardly be able to detect the Ising fixed point.

2.2 Classical solutions in finite volume

Before discussing the kink solution on the cylinder, it is worth briefly recalling that the DHN method can be also applied to the constant solutions describing the vacua in the periodic sector of the theory. In particular, for the potential (2.4) we have

$$\begin{aligned} \phi_{cl}^{vac}(x) &\equiv (\pm) \frac{m}{\sqrt{\lambda}} , \\ \omega_n^{vac} &= \sqrt{2m^2 + \left(\frac{2n\pi}{R}\right)^2} , \quad n = 0, \pm 1, \pm 2 \dots \end{aligned} \quad (2.17)$$

Therefore, according to (2.2), the smallest mass gap in the system, i.e. the difference between the first excited state and the ground state, is given by:

$$E_1(R) - E_0(R) = \omega_0^{vac}(R) \equiv \sqrt{2} m . \quad (2.18)$$

This quantity, which is related to the inverse correlation length ξ^{-1} on a finite size [3, 25, 14], is the one that has to be used³ in the definition of the scaling variable

$$r \equiv m R . \quad (2.19)$$

If we now want to describe a kink on a cylinder of circumference R , we have to look for a solution of eq. (2.6) satisfying the antiperiodic boundary conditions (2.5). This can be found for $1 < \bar{\phi}_0 < \sqrt{2}$, and it is expressed as

$$\bar{\phi}_{cl}(\bar{x}) = \sqrt{2 - \bar{\phi}_0^2} \operatorname{sn} \left(\frac{\bar{\phi}_0}{\sqrt{2}} (\bar{x} - \bar{x}_0) , k \right) , \quad (2.20)$$

where $\operatorname{sn}(u, k)$ is the Jacobi elliptic function with modulus $k^2 = \frac{2}{\bar{\phi}_0^2} - 1$ and period $4\mathbf{K}(k^2)$, where $\mathbf{K}(k^2)$ is the complete elliptic integral of the first kind (see Appendix A for the definitions and properties of elliptic integrals and Jacobi elliptic functions). As shown in Fig. 2, the classical solution (2.20) oscillates between the values $-\sqrt{2 - \bar{\phi}_0^2}$ and $\sqrt{2 - \bar{\phi}_0^2}$, and the boundary conditions (2.5) are satisfied by relating the elliptic modulus to the size of the system as

$$mR = \sqrt{1 + k^2} \, 2 \mathbf{K}(k^2) . \quad (2.21)$$

³Up to inessential numerical constants which we fix here to $1/\sqrt{2}$ for later convenience.

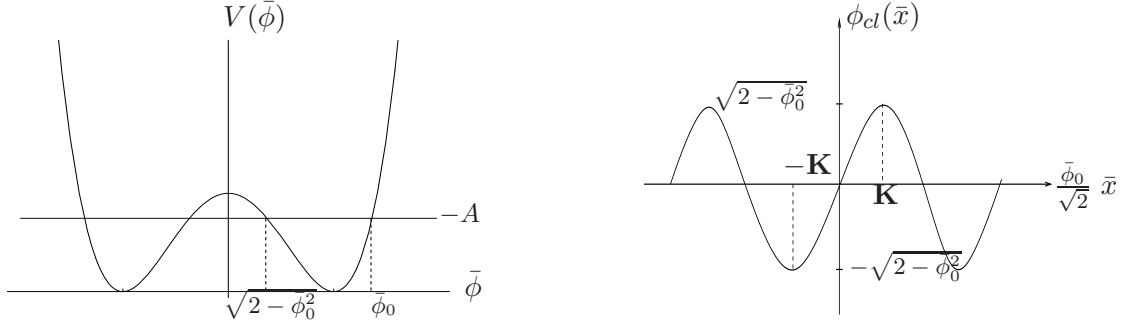


Figure 2: Potential (2.4) and finite-volume kink (2.20) with $x_0 = 0$.

As expected, (2.20) goes to the infinite-volume kink (2.8) for $k \rightarrow 1$ (i.e. $\bar{\phi}_0 \rightarrow 1$), which corresponds to the infrared limit $mR \rightarrow \infty$. In the complementary limit $k \rightarrow 0$ (i.e. $\bar{\phi}_0 \rightarrow \sqrt{2}$), which corresponds to $mR \rightarrow \pi$, the kink (2.20) tends to the constant solution

$$\phi_{cl}(x) \equiv 0, \quad (2.22)$$

which identically satisfies the antiperiodic b.c. (2.5) and can be used, therefore, as the background field configuration in the interval $0 < mR < \pi$. The choice of the background

$$\phi_{cl}(x) = \begin{cases} \sqrt{2 - \bar{\phi}_0^2} \operatorname{sn}\left(\frac{\bar{\phi}_0}{\sqrt{2}}(\bar{x} - \bar{x}_0), k\right) & \text{for } mR > \pi \\ 0 & \text{for } mR < \pi \end{cases} \quad (2.23)$$

will be fully motivated in the following, after the discussion of the stability frequencies related to the classical solutions (2.20) and (2.22).

The classical energy of the kink (2.23) is given by

$$\mathcal{E}_{cl}(R) = \begin{cases} \frac{m^3}{6\lambda} \frac{1}{(1+k^2)^{3/2}} \{3k^4 \mathbf{K}(k^2) + 2k^2 [\mathbf{K}(k^2) + 4\mathbf{E}(k^2)] + 8\mathbf{E}(k^2) - 5\mathbf{K}(k^2)\} & \text{for } mR > \pi \\ \frac{m^3}{4\lambda} mR & \text{for } mR < \pi \end{cases}, \quad (2.24)$$

and it is plotted in Fig. 3. From the analytic knowledge of this quantity, we can immediately extract some important scattering data of the non-integrable ϕ^4 theory. In fact, the leading term in the kink mass is given by the classical energy, expressed for generic R by (2.24). It is easy to see that for $R \rightarrow \infty$ the energy indeed tends to the infinite-volume limit $\mathcal{E}_{cl}(R) \rightarrow \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda}$. From its asymptotic expansion for large R , we can also obtain the leading order of the kink mass correction in finite volume, and compare it with Lüscher's theory [11, 12]. Taking into account the $k \rightarrow 1$ ($k' \rightarrow 0$) expansions of \mathbf{E} and \mathbf{K} (see Appendix A) and noting from (2.21) that

$$e^{-\sqrt{2}mR} = \frac{1}{256}(k')^4 + \dots,$$

we derive the following asymptotic expansion of \mathcal{E}_{cl} for large R :

$$\mathcal{E}_{cl}(R) = \mathcal{E}_{cl}(\infty) - 8\sqrt{2} \frac{m^3}{\lambda} e^{-\sqrt{2}mR} + O(e^{-2\sqrt{2}mR}). \quad (2.25)$$

The counterpart of this leading-order behavior in Lüscher's theory is given by

$$M_k(R) - M_k(\infty) = -m_b R_{k k b} e^{-m_b R}, \quad (2.26)$$

where the index k refers to the kink, and the index b refers to the elementary meson (with mass $m_b = \sqrt{2}m$), which can be seen as a kink-antikink bound state with S -matrix residue $R_{k k b}$. From the comparison between (2.25) and (2.26) we finally extract the leading semiclassical expression for the residue of this 3-particle process

$$R_{k k b} = 8 \frac{m^2}{\lambda}, \quad (2.27)$$

and therefore the 3-particle coupling⁴

$$\Gamma_{k \bar{k} b} = 2\sqrt{2} \frac{m}{\sqrt{\lambda}}. \quad (2.28)$$

This quantity is of particular interest, since the non-integrability of the ϕ^4 theory prevents the knowledge of its exact S -matrix. In the different context of infinite volume form factors, in [10] we proposed another way of extracting this coupling, i.e. by looking at the residue of the kink-antikink form factor in infinite volume, and the result obtained in [10] is consistently equal to (2.28).

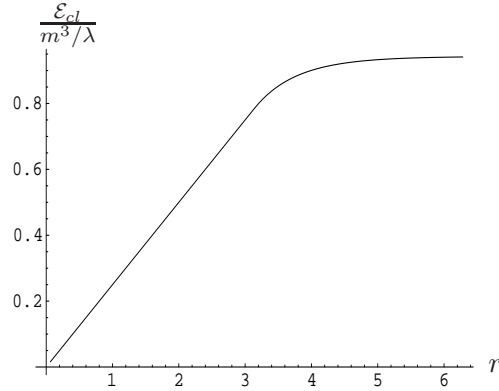


Figure 3: Classical energy (2.24)

2.3 Semiclassical scaling functions

The stability equation (2.1) around the background (2.20) takes the form

$$\left\{ \frac{d^2}{d\bar{x}^2} + \bar{\omega}^2 + 1 - 3(2 - \bar{\phi}_0^2) \operatorname{sn}^2 \left(\frac{\bar{\phi}_0}{\sqrt{2}} \bar{x}, k^2 \right) \right\} \bar{\eta}(\bar{x}) = 0, \quad (2.29)$$

where $\bar{\omega} = \omega/m$, and it can be reduced to the Lamé equation with $N = 2$ (see Appendix B). The allowed and forbidden bands, with corresponding values of the Floquet exponent, are shown in Fig. 4.

⁴Crossing symmetry implies the equality $R_{k \bar{k} b} = R_{k k b}$.

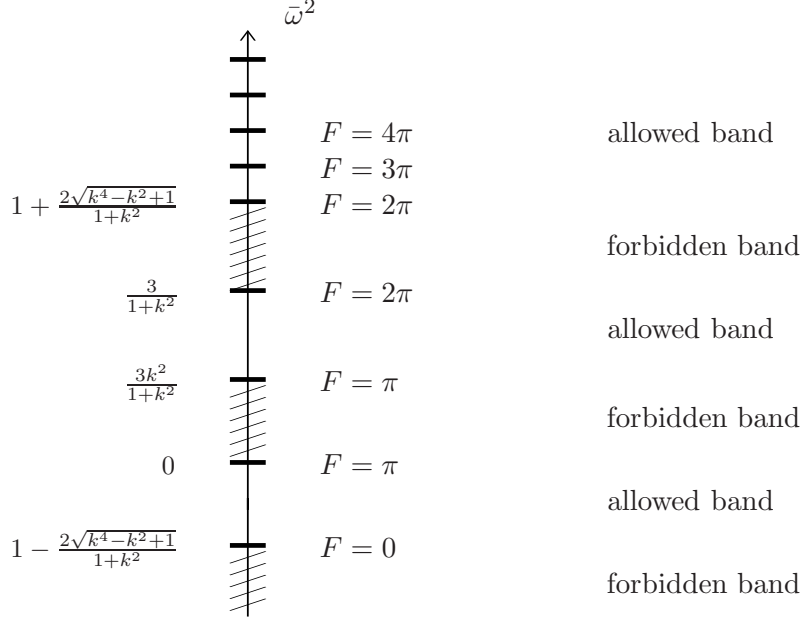


Figure 4: Spectrum of eq. (2.29)

The boundary conditions (2.5) translate into the requirement of antiperiodicity for the fluctuation η

$$\eta(x+R) = -\eta(x),$$

which selects the values of $\bar{\omega}^2$ for which the Floquet exponent is an odd multiple of π . These eigenvalues are the zero mode

$$\bar{\omega}_0^2 = 0, \quad (2.30)$$

the discrete value

$$\bar{\omega}_1^2 = \frac{3k^2}{1+k^2}, \quad (2.31)$$

and the infinite series of points (with multiplicity 2) inside the highest band

$$\bar{\omega}_n^2 \equiv 1 - \frac{3}{1+k^2} [\mathcal{P}(a_n) + \mathcal{P}(b_n)], \quad (2.32)$$

with a_n, b_n constrained by

$$\begin{cases} F = 2i \{ \mathbf{K}[\zeta(a_n) + \zeta(b_n)] - (a_n + b_n) \zeta(\mathbf{K}) \} = (2n-1)\pi \\ \mathcal{P}'(a_n) + \mathcal{P}'(b_n) = 0 \end{cases}, \quad n = 2, 3, \dots \quad (2.33)$$

In the IR limit ($k \rightarrow 1$) this spectrum goes to the one related to the standard background (2.8). In fact, the allowed band $1 - \frac{2\sqrt{k^4-k^2+1}}{1+k^2} < \bar{\omega}^2 < 0$ shrinks to the eigenvalue $\bar{\omega}_0^2 = 0$, the other band $\frac{3k^2}{1+k^2} < \bar{\omega}^2 < \frac{3}{1+k^2}$ shrinks to $\bar{\omega}_1^2 = \frac{3}{2}$, and finally $\bar{\omega}^2 > 1 + \frac{2\sqrt{k^4-k^2+1}}{1+k^2}$ goes to the continuous part of the spectrum $\bar{\omega}_q^2 = 2 + \frac{1}{2}q^2$.

In order to complete the spectrum also at values $mR < \pi$, we have to put together the frequencies (2.31) and (2.32) with the ones obtained by quantizing the constant solution (2.22).

We therefore obtain⁵

$$\bar{\omega}_1^2 = \begin{cases} \frac{3k^2}{1+k^2} & \text{for } mR > \pi \\ -1 + \frac{\pi^2}{m^2 R^2} & \text{for } mR < \pi \end{cases}, \quad (2.34)$$

and

$$\bar{\omega}_n^2 = \begin{cases} 1 - \frac{3}{1+k^2} [\mathcal{P}(a_n) + \mathcal{P}(b_n)] & \text{for } mR > \pi \\ -1 + (2n-1)^2 \frac{\pi^2}{m^2 R^2} & \text{for } mR < \pi \end{cases}. \quad (2.35)$$

With the explicit knowledge of the stability frequencies, and in particular of the first one, plotted in Fig. 5, we can now understand the physical meaning of the point $mR = \pi$. This corresponds, in fact, to the limit $k \rightarrow 0$ and this is the value below which the analytic continuation of the classical background (2.20) becomes imaginary. Correspondingly, the first frequency square ω_1^2 tends to zero, and its continuation would become negative, signaling an instability of the solution. At the same time, the constant background (2.22) is stable just up to the point $mR = \pi$, as it can be easily seen from Fig. 5.

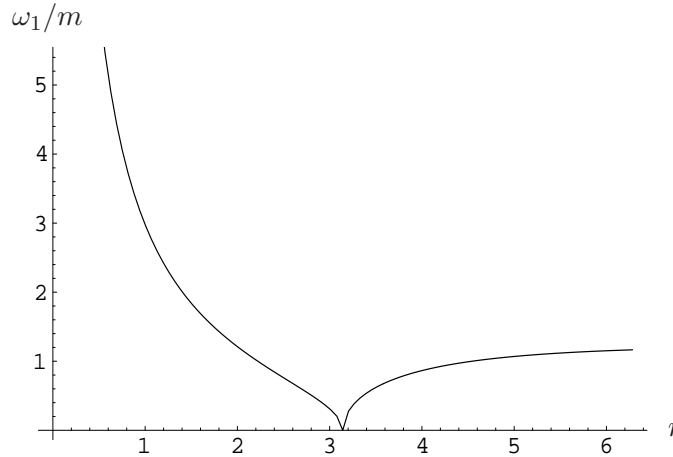


Figure 5: The first level defined in (2.34)

Figure 6 shows the plots, for generic values of r in (2.19), of the first few frequencies given by (2.34) and (2.35), which represent the energies of the excited states with respect to their ground state $E_0(R)$.

We have now all data to write the ground state energy in the kink sector, which is defined in analogy with the infinite volume case (2.12) as

$$E_0(R) = \mathcal{E}_{cl}(R) + \frac{1}{2} \sum_i \omega_i(R) + C.T. - \frac{1}{2} \sum_{n=-\infty}^{\infty} \omega_n^{\text{vac}}(R), \quad (2.36)$$

⁵To be precise, notice that the eigenvalue $\bar{\omega}_1^2 = -1 + \frac{\pi^2}{m^2 R^2}$ is double, and at $mR = \pi$ it splits into the two simple eigenvalues (2.30) and (2.31).

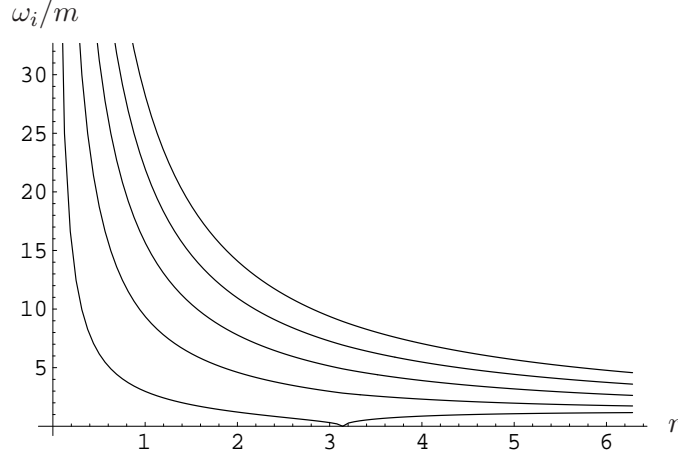


Figure 6: The first few levels defined in (2.34) and (2.35)

where the frequencies ω_i are defined in (2.34) and (2.35), and the mass counterterm is defined as

$$C.T. = -\frac{\delta m^2}{2} \int_{-R/2}^{R/2} dx \left[\left(\phi_{cl}^{\text{kink}}(x) \right)^2 - \frac{m^2}{\lambda} \right] ,$$

with

$$\delta m^2 = \frac{3}{4\pi} \lambda \frac{2\pi}{R} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^{\text{vac}}} , \quad \omega_n^{\text{vac}}(R) = \sqrt{2m^2 + \left(\frac{2n\pi}{R} \right)^2} .$$

A more transparent expression for the ground state energy (2.36), which explicitly shows the cancellation of the divergencies present in each term separately, can be obtained by expanding all quantities around some specific value of r . In particular, in the limits of large or small r one can extract the asymptotic IR and UV data of the theory. We have already seen in Sect. 2.2 how the large- r expansion of the classical energy correctly encodes the scattering data of the infinite volume theory, and we will now study the UV limit $r \rightarrow 0$, in which we can extract some conformal data related to the theory in exam. Furthermore, in Appendix C we perform the expansion around the point $r = \pi$, where it is possible to see how the divergencies cancel in a more subtle way.

The small- r expansion of (2.36) is easily obtained to be

$$\frac{E_0(R)}{m} = \frac{2\pi}{r} \left[\sum_{n=1}^{\infty} \left(n - \frac{1}{2} \right) - \sum_{n=1}^{\infty} n \right] - \frac{1}{4\sqrt{2}} + \frac{r}{2\pi} \left[\frac{\pi}{2} \frac{m^2}{\lambda} - \sum_{n=1}^{\infty} \frac{1}{2n-1} + \sum_{n=1}^{\infty} \frac{1}{2n} \right] + \dots . \quad (2.37)$$

The individually divergent series present in (2.37) combine to give a finite result, in virtue of

the relations

$$\begin{aligned} \sum_{n=1}^{\infty} (2n-1) - \sum_{n=1}^{\infty} (2n) &= 2 [\zeta(-1, 1/2) - \zeta(-1)] = 2 \left[\frac{1}{24} + \frac{1}{12} \right] , \\ \sum_{n=1}^{\infty} \frac{1}{(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{(2n)} &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2 . \end{aligned}$$

The UV behaviour for $r \rightarrow 0$ of the ground state energy $E_0(R)$ of a given off-critical theory is related to the Conformal Field Theory (CFT) data (h, \bar{h}, c) of the corresponding critical theory and to the bulk energy term as

$$E_0(R) \simeq \frac{2\pi}{R} \left(h + \bar{h} - \frac{c}{12} \right) + \mathcal{B} R + \dots \quad (2.38)$$

where c is the central charge, $h + \bar{h}$ is the lowest anomalous dimension in a given sector of the theory and \mathcal{B} the bulk coefficient. Therefore, we estimate the semiclassical bulk term to be given by

$$\mathcal{B} = m^2 \left(\frac{1}{4} \frac{m^2}{\lambda} - \frac{\log 2}{2\pi} \right) .$$

Furthermore, our result for the leading semiclassical term in the anomalous dimension is ⁶

$$h + \bar{h} = \frac{1}{8} . \quad (2.39)$$

As it is fully discussed in Sect. 3, this result agrees with the CFT prediction.

Finally, again in accordance with the CFT expectation, the excited levels are given by

$$E_{k_n}(R) \simeq \frac{2\pi}{R} \left[\frac{1}{8} + \sum_n k_n \left(n - \frac{1}{2} \right) \right] + \left[\mathcal{B} - \frac{m^2}{2\pi} \sum_n \frac{k_n}{2n-1} \right] R + \dots \quad (2.40)$$

2.4 Other interpretations of the classical solution

In concluding this Section, it is worth to comment how the classical solution (2.20) has been studied in the literature either in different contexts, or in the same as ours but along different lines of interpretation.

In fact, this kind of background, regarded however as a time-dependent solution in zero space dimensions, has been proposed in [16] to describe dominating contributions to the partition function at finite temperature T , i.e. when the Euclidean time variable is compactified on a circumference $\beta = \frac{1}{k_B T}$ with *periodic* b.c.. In this case, the finite value of T which corresponds to $k = 0$ is naturally interpreted as a limiting temperature, above which no periodic solutions exist.

Moreover, the background (2.20) has also been studied in [17, 18] as a static classical solution on a cylindrical geometry. In these works, however, *periodic* b.c. are considered, and the size of the system is related to the elliptic modulus as

$$mR = \sqrt{1 + k^2} \, 4N \mathbf{K}(k^2) , \quad \text{with } N \in \mathbb{N} .$$

⁶Notice that the central charge contribution $-c/12$ is absent in (2.37), because we are subtracting the ground state energies of kink and vacuum sector, which both have the same central charge $c = 1$.

This choice, which corresponds to considering the solution as a train of N kinks and N antikinks, implies the selection of N distinct eigenvalues with $\omega_n^2 < 0$ in the spectrum of eq. (2.29). Their imaginary contributions to the energy levels indicate the instability of the considered background, which is explained in [18] by noting that in the $k \rightarrow 1$ ($R \rightarrow \infty$) limit the solution tends to a single kink, instead of keeping its periodic nature of a train of kinks and antikinks. All this reflects the ambiguity present in the definition of the size of the system R in terms of the elliptic modulus k , simply due to the periodicity of the Jacobi function $\text{sn}(u, k)$, and correspondingly in the interpretation of the solution for a chosen definition of R . However, choosing (2.21), i.e. *antiperiodic* b.c., the infinite volume limit is smoothly recovered as $k \rightarrow 1$, and the corresponding single kink solution is stable. It is then natural to expect that for any value of R of the finite system, also time-dependent solutions exist, which describe multikink or kink–antikink configurations. Such solutions can be quantized in finite volume as well, although this is a subject that is out of the scope of the present paper.

Finally, in the recent paper [19] the orbifold geometry S^1/\mathbb{Z}_2 is considered, instead of the circle, for the worldsheet space coordinate x , and a classical background very similar to (2.23) is introduced. The analogy with our case, however, is only apparent. In fact, due to the absence of translational invariance, on the orbifold the kink and the antikink have to be considered as two distinct degenerate solutions, suggesting therefore a phase transition at $mR = \pi$. In our case, on the contrary, the lowest energy level is never degenerate, consistently with the fact that the behavior of the scaling functions at $mR = \pi$ does not hint at any underlying conformal field theory. The discontinuity of the derivative of ω_1 at $mR = \pi$ should be then interpreted as just an effect of the semiclassical approximation.

3 Kink–creating operators in Landau–Ginsburg models

As it is well known, starting from $c = 1$ CFT in two dimensions and adding to its Lagrangian different relevant operators with an appropriate choice of the coupling constants, one can construct many integrable and non-integrable 2D massive QFT’s having degenerate vacua [4]. They can be classified according to the symmetries preserved by the perturbation. For instance, SG and Double SG models are examples of $Z \otimes Z_2$ -invariant theories (i.e. $\phi \rightarrow \pm\phi + 2\pi n$), while LG models are Z_2 -invariant (i.e. $\phi \rightarrow -\phi$) ones. The common feature of all these models are the non-perturbative topologically stable classical solutions (solitons or kinks) interpolating between two vacua. In the quantum theory they give rise to specific “strong coupling” particles, carrying topological (Z or/and Z_2) charges and representing an important part of their IR spectrum. The description of the finite volume spectrum (on the cylinder) of these models therefore requires both the construction of the finite volume counterparts of such topological solutions and the identification of the quantum states related to them. An important consistency check for the finite volume spectrum is provided by its UV and IR limits (in the scaling variable mR) that should reproduce the CFT and the massive model spectra correspondingly. In order to understand the flow between the UV theory to the IR one, i.e. the relation between the CFT

space of states (and the corresponding field operators) and the infinite volume (massive) particle space of states, it is also necessary to recognize the states (and operators) that describe the UV limits of such solitons and kinks in the $c = 1$ CFT. The soliton (and kink) creating operators are non-local functionals of the field ϕ that satisfy the following requirements:

(a) to carry (Z or Z_2) topological charges ± 1 or equivalently to produce specific b.c.'s ⁷ for ϕ ,

$$\phi(ze^{2i\pi}, \bar{z}e^{-2i\pi}) = \phi(z, \bar{z}) + 2\pi n\mathcal{R} \quad n = \pm 1 \quad (3.1)$$

for solitons (where \mathcal{R} is the compactification radius of ϕ , say $\mathcal{R} = \beta^{-1}$ for SG), and

$$\phi(ze^{2i\pi}, \bar{z}e^{-2i\pi}) = -\phi(z, \bar{z}) \quad (3.2)$$

for the (Z_2) kinks.

(b) to be local with respect to the perturbation (i.e., $V_l(\phi) = \sum_{k=1}^l \lambda_k \phi^{2k-2}$ for the LG models) or/and to the corresponding energy density operator in order to have well defined off-critical properties.

Before discussing the construction of the kink-creating operators for the LG models (1.1), it is worthwhile to remind how the soliton operators are derived in the case of SG model [20, 21]. As it well known [22], the primary fields in the untwisted (“winding”) sector⁸ of the (compact) $c = 1$ Gaussian CFT are represented by the following *discrete* set of vertex operators

$$V_{n,s}(z, \bar{z}) =: \exp(ip\phi + i\bar{p}\tilde{\phi}) :$$

with

$$p = \frac{s}{\mathcal{R}} \quad , \quad \bar{p} = 2\pi g n \mathcal{R} \quad , \quad n, s = 0, \pm 1, \pm 2, \dots$$

Their “chiral” dimensions⁹ are given by $h = \frac{(p+\bar{p})^2}{8\pi g}$ and $\bar{h} = \frac{(p-\bar{p})^2}{8\pi g}$ and therefore they have spin $s = h - \bar{h}$ and dimension $\Delta = h + \bar{h}$. We have introduced the free fields $\varphi(z)$ and $\bar{\varphi}(\bar{z})$ such that $\phi = \varphi(z) + \bar{\varphi}(\bar{z})$ and its dual is $\tilde{\phi} = \varphi(z) - \bar{\varphi}(\bar{z})$. They take values on the circle S_1 with radius $\mathcal{R} = \frac{1}{\beta}$ and their correlation functions have the form:

$$\langle \varphi(z)\varphi(w) \rangle = -\frac{1}{4\pi g} \ln(z-w), \quad \langle \bar{\varphi}(\bar{z})\bar{\varphi}(\bar{w}) \rangle = -\frac{1}{4\pi g} \ln(\bar{z}-\bar{w}) \quad (3.3)$$

As one can easily verify from the OPE

$$\phi(z, \bar{z})V_{n,s}(0,0) = -\frac{i}{4\pi g} \left(\bar{p} \ln\left(\frac{z}{\bar{z}}\right) + p \ln(z\bar{z}) \right) V_{n,s}(0,0) + \dots \quad (3.4)$$

the vertex operators $V_{n,s}$ for $n = \pm 1$ and for arbitrary spin s , create the Z -type b.c.'s (3.1) (in fact one can take, say $s = 0$ or $s = \pm 1$, since the only $\tilde{\phi}$ contribution is relevant). They are also

⁷the relation between the z and \bar{z} coordinates used in this section and the x and t used in all the others is the standard plane to cylinder one, i.e. $z = e^{\frac{i}{R}(x+t)}$ and $\bar{z} = e^{-\frac{i}{R}(x-t)}$.

⁸defined by the condition that the chiral $U(1)$ currents $I(z) = \partial\varphi(z)$ and $\bar{I}(\bar{z}) = \bar{\partial}\varphi(\bar{z})$ are single valued.

⁹Note that we have introduced arbitrary normalization constant g in the action $A_{gauss} = \frac{g}{2} \int d^2x (\partial_\mu \phi) (\partial^\mu \phi)$ and as a consequence the the chiral component of the stress-tensor $T(z, \bar{z})$ is given by $T = -2\pi g : (\partial\phi)^2 :$. The standard CFT normalization is $g = \frac{1}{2\pi}$, but we shall often use $g = 1$.

local with respect to the SG potential $V_{SG} = \frac{m^2}{\beta^2} \cos(\beta\phi)$ as it follows from their OPE's (with $\mathcal{R} = \frac{1}{\beta}$)

$$\cos(\beta\phi(z, \bar{z}))V_{n,s}(0,0) = \frac{1}{2} \left(\frac{z}{\bar{z}}\right)^{\frac{\bar{p}\beta}{4\pi g}} (z\bar{z})^{\frac{p\beta}{4\pi g}} V_{n,s+1}(0,0) + \frac{1}{2} \left(\frac{z}{\bar{z}}\right)^{-\frac{\bar{p}\beta}{4\pi g}} (z\bar{z})^{-\frac{p\beta}{4\pi g}} V_{n,s-1}(0,0) + \dots \quad (3.5)$$

i.e. we have no changes under the transformation $(ze^{2i\pi}, \bar{z}e^{-2i\pi})$ to (z, \bar{z}) , since $\bar{p} = \frac{2\pi g n}{\beta}$ and $\frac{\bar{p}\beta}{4\pi g} = \frac{n}{2}$. Therefore for $n = \pm 1$ they represent the one soliton-creating operators. The operators with $n \geq 2$ create multi-soliton states. It should be noted that in the perturbed CFT (i.e. in SG theory) the dual field $\tilde{\phi}$ is nonlocal in terms of the SG field ϕ , i.e. we have $\tilde{\phi}(x, t) = \int_{-\infty}^x dy \partial_y \phi(x, y)$. The Z topological (i.e. soliton) charge Q is defined by the eigenvalues of the well known SG charge operator

$$Q = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi(x, t). \quad (3.6)$$

In order to describe the operators that create Z_2 -type (antiperiodic) b.c.'s (3.2) for the SG field ϕ we have to consider the twisted sector of the $c = 1$ CFT. It is defined (see ref. [23]) by the condition that the chiral $U(1)$ currents $I(z) = \partial\varphi(z)$ and $\bar{I}(\bar{z}) = \bar{\partial}\varphi(\bar{z})$ are double valued, i.e. their mode expansions contain only half-integer modes

$$I(z) = \sum_{m=-\infty}^{\infty} I_{m-\frac{1}{2}} z^{-m-\frac{1}{2}}, \quad \bar{I}(\bar{z}) = \sum_{m=-\infty}^{\infty} \bar{I}_{m-\frac{1}{2}} \bar{z}^{-m-\frac{1}{2}} \quad (3.7)$$

where the modes $I_{m-\frac{1}{2}}$ (and $\bar{I}_{m-\frac{1}{2}}$) satisfy the following Heisenberg type algebra:

$$[I_{m-\frac{1}{2}}, I_{l-\frac{1}{2}}] = \frac{m-\frac{1}{2}}{2} \delta_{m+l}, \quad [I_{m-\frac{1}{2}}, \bar{I}_{l-\frac{1}{2}}] = 0 \quad (3.8)$$

The primary fields in this sector $\mu_{k,\bar{k}}^{\pm}$, i.e.

$$I_{m+\frac{1}{2}} \mu_{k,\bar{k}}^{\pm} = 0 \quad \bar{I}_{m+\frac{1}{2}} \mu_{k,\bar{k}}^{\pm} = 0, \quad m, \bar{k}, k = 0, 1, 2, \dots \quad (3.9)$$

have “chiral” dimensions $h_k = \frac{(2k+1)^2}{16}$ and $\bar{h}_{\bar{k}} = \frac{(2\bar{k}+1)^2}{16}$ and the allowed spins are given by $s = 0, \pm\frac{1}{2}$. As one can see from the OPE

$$\phi(z, \bar{z}) \mu_0^{\pm}(0,0) = \sqrt{z} \mu_1^{\pm}(0,0) + \sqrt{\bar{z}} \bar{\mu}_1^{\pm}(0,0) + \dots \quad (3.10)$$

the fields $\mu_{0,0}^{\pm}(0,0) = \mu_0^{\pm}$ (of lowest dimension $h + \bar{h} = \frac{1}{8}$ and spin $s = 0$), called disorder (or spin) fields, create branch cut singularity for ϕ and thus reproduces the Z_2 -type b.c.'s (3.2). Their locality with respect to $\cos(\beta\phi)$ is a consequence of the OPE (3.10) and of the following correlation function

$$\begin{aligned} & < \mu_0^-(\infty, \infty) e^{i\alpha\phi(w, \bar{w})} \cos(\beta\phi(z, \bar{z})) \mu_0^+(0,0) > = \\ & = \frac{C_{+-}}{2} \left[\left(\frac{(\sqrt{w}-\sqrt{z})(\sqrt{\bar{w}}-\sqrt{\bar{z}})}{(\sqrt{w}+\sqrt{z})(\sqrt{\bar{w}}+\sqrt{\bar{z}})} \right)^{\frac{\alpha\beta}{4\pi g}} + \left(\frac{(\sqrt{w}-\sqrt{z})(\sqrt{\bar{w}}-\sqrt{\bar{z}})}{(\sqrt{w}+\sqrt{z})(\sqrt{\bar{w}}+\sqrt{\bar{z}})} \right)^{-\frac{\alpha\beta}{4\pi g}} \right]. \end{aligned} \quad (3.11)$$

Note that the current $I(z)$ does not have zero mode in the twisted sector and therefore the fields $\mu_{k,\bar{k}}^\pm$ do not carry $U(1)$ (and Z), but only Z_2 charge. All these properties of the disorder field $\mu_0^\pm(0,0)$ lead to the conclusion that it represents the kink-creating operator. It should be mentioned that the field ϕ in this case takes its values on the orbifold $\frac{S_1}{Z_2}$ and, as usually, the two disorder fields $\mu_0^\pm(0,0)$ are related to the two fixed points $\phi = 0$ and $\phi = \pi\mathcal{R}$ ([15]). As we shall show in Sect. 4, in finite volume one can have both the quasiperiodic (soliton type) and antiperiodic (kink type) solutions and states, which however in the IR (infinite volume) SG theory are related to the same soliton (and anti-soliton) states.

The description of the kink-creating operators in the LG models is quite similar to the one of the SG model. The main difference is that the field ϕ is no longer compactified, i.e. it lives now on the orbifolded line $\frac{R^{(1)}}{Z_2}$. The untwisted (i.e. Z_2 -even) sector of the corresponding (noncompact) $c = 1$ CFT contains two *continuous* parameters (q, \bar{q}) family of vertex operators $V_{q,\bar{q}} =: \exp(iq\phi + i\bar{q}\tilde{\phi})$: of “chiral” dimensions $h = \frac{(q+\bar{q})^2}{8\pi g}$ and $\bar{h} = \frac{(q-\bar{q})^2}{8\pi g}$. As in SG case the operators with $\bar{q} \neq 0$ produce certain nontrivial b.c.’s for ϕ , but with continuous $U(1)$ charge. As expected, there is not a properly defined Z topological charge in this case. Such operators are also non-local with respect to the LG potential (1.1) as it can be seen from the OPE’s, say

$$: \phi(z, \bar{z})^k :: e^{i\bar{q}\tilde{\phi}(0,0)} : = : \left(-\frac{i\bar{q}}{4\pi g} \ln \frac{z}{\bar{z}} + \phi(0,0) \right)^k e^{i\bar{q}\tilde{\phi}(0,0)} : + \dots \quad (3.12)$$

Therefore they cannot represent kink-creating operators. The structure of the twisted sector of this noncompact $c = 1$ CFT is quite similar to the one considered in the context of the SG (i.e. $\cos(\beta\phi)$) perturbation above. Since in the orbifold line (as well as in orbifold finite interval) we have only one fixed point $\phi = 0$, we have correspondingly only one disorder field μ_0 of dimension $1/8$ and spin zero. As in the SG case, the field μ_0 produces branch cut in the OPE with ϕ and so, it implements the Z_2 -type (antiperiodic) b.c.’s (3.2). In order to check whether it is local with respect to the LG potential let’s consider its correlation functions

$$< \mu_0(\infty, \infty) e^{i\alpha\phi(w, \bar{w})} e^{i\gamma\phi(z, \bar{z})} \mu_0(0,0) > = C_0 \left(\frac{\sqrt{w}-\sqrt{z}}{\sqrt{w}+\sqrt{z}} \right)^{\frac{\alpha\gamma}{4\pi g}} \left(\frac{\sqrt{\bar{w}}-\sqrt{\bar{z}}}{\sqrt{\bar{w}}+\sqrt{\bar{z}}} \right)^{\frac{\alpha\gamma}{4\pi g}}, \quad (3.13)$$

$$< \mu_0(\infty, \infty) e^{i\alpha\phi(w, \bar{w})} : \phi(z, \bar{z})^k : \mu_0(0,0) > = C_0 (-i)^k \left(\frac{\alpha}{4\pi g} \ln \left(\frac{\sqrt{w}-\sqrt{z}}{\sqrt{w}+\sqrt{z}} \right) \left(\frac{\sqrt{\bar{w}}-\sqrt{\bar{z}}}{\sqrt{\bar{w}}+\sqrt{\bar{z}}} \right) \right)^k \quad (3.14)$$

These can be derived from the φ mode expansion $\varphi(z) = \sum_{m=-\infty}^{\infty} \frac{I_{m-\frac{1}{2}}}{\frac{1}{2}-m} z^{-m+\frac{1}{2}}$, the algebra (3.8) of its modes and the properties (3.9) of the disorder field μ_0 . It is now easy to see that each (linear) combination of even powers of the field ϕ is local with respect to μ_0 , i.e. it does not change under the transformation $(ze^{2i\pi}, \bar{z}e^{-2i\pi})$ to (z, \bar{z}) . It becomes clear from this discussion that the only field that can create Z_2 -kinks in the LG models is then the disorder field μ_0 . In the “broken phase” ϕ^4 model (2.4) we have only one kink interpolating between the two minima

of the potential. In the symmetric type LG potentials, as for example

$$\begin{aligned} V_l^{\text{odd}} &= \frac{1}{2} \prod_{k=1}^{\frac{l-1}{2}} (\phi^2 - a_k^2)^2 & \text{for } l = 3, 5, \dots \\ V_l^{\text{even}} &= \frac{1}{2} \phi^2 \prod_{k=1}^{\frac{l-2}{2}} (\phi^2 - a_k^2)^2 & \text{for } l = 4, 6, \dots \end{aligned} \quad (3.15)$$

we have instead a finite number of l degenerate vacua and therefore different kinks relating each two consecutive vacua. An important question is: how to distinguish them in a finite volume? Moreover, in the CFT language, what are the operators which create such kinks?

To answer such questions, observe that the minima of these potentials are at the points $\phi_k = \pm a_k$ ($k = 1, 2, \dots, \frac{l-1}{2}$ for l odd) and since we consider $a_1 > a_2 > \dots$ the kinks are interpolating between ϕ_1 and ϕ_2 , etc. and not, as in the ϕ^4 case, between $\pm\phi_0$. Therefore the analog of the antiperiodic b.c.'s (3.2) for the case of many degenerate vacua is given by

$$\phi(ze^{2i\pi}, \bar{z}e^{-2i\pi}) = a_k + a_{k+1} - \phi(z, \bar{z}), \quad (3.16)$$

i.e. we have different b.c.'s for each kink. Indeed one can reduce such b.c.'s to the standard ones (3.2) by introducing the “shifted” fields and the analog of the antiperiodic b.c.'s (3.2) in the case of many degenerate vacua is given by

$$\Phi_k(z, \bar{z}) = \phi(z, \bar{z}) - \frac{(a_k + a_{k+1})}{2} \quad (3.17)$$

In this scheme, however, the new fields have different vacua expectation values. Since (different) orbifolds based on (3.16) have different fixed points, one can formally prescribe to each such point one disorder field $\mu^{(k)}(z, \bar{z})$. As we shall see on the example of the ϕ^6 model in Sect. 5 below, although all these kinks have coinciding UV data, their finite volume scaling functions are however different, with different bulk coefficients etc.

4 Sine-Gordon model with antiperiodic b.c.

In the light of the discussion of kink-creating operators presented in Sect. 3, it is worth to illustrate in more detail the interesting case of the Sine-Gordon model, where both kinds of kink exist. This fact can be easily understood in the framework of the correspondence between Sine-Gordon and Thirring models. In fact, the Sine-Gordon solitons are identified with the Thirring fermions, for which two types of boundary conditions (periodic and antiperiodic) can be naturally imposed in a finite volume.

The Euler-Lagrange equation for static backgrounds in the Sine-Gordon model take the form

$$\frac{1}{2} \left(\frac{\partial \phi_{cl}}{\partial x} \right)^2 = \frac{m^2}{\beta^2} (1 - \cos \beta \phi_{cl} + A) \quad , \quad (4.1)$$

and it admits three kinds of solution, depending on the sign of the constant A . The simplest corresponds to $A = 0$ and it describes the standard kink in infinite volume:

$$\phi_{cl}^0(x) = \frac{4}{\beta} \arctan e^{m(x-x_0)} . \quad (4.2)$$

The other two solutions, relative to the case $A \neq 0$, can be expressed in terms of Jacobi elliptic functions [24], defined in Appendix A. In particular, for $A > 0$ we have

$$\phi_{cl}^+(x) = \frac{\pi}{\beta} + \frac{2}{\beta} \operatorname{am} \left(\frac{m(x-x_0)}{k}, k \right) , \quad k^2 = \frac{2}{2+A} , \quad (4.3)$$

which has the monotonic and unbounded behaviour in terms of the real variable $u^+ = \frac{m(x-x_0)}{k}$ shown in Fig. 7. For $-2 < A < 0$, the solution is given instead by

$$\phi_{cl}^-(x) = \frac{2}{\beta} \arccos [k \operatorname{sn}(m(x-x_0), k)] , \quad k^2 = 1 + \frac{A}{2} , \quad (4.4)$$

and it oscillates in the real variable $u^- = m(x-x_0)$ between the k -dependent values $\tilde{\phi}$ and $\frac{2\pi}{\beta} - \tilde{\phi}$ (see Fig. 7).

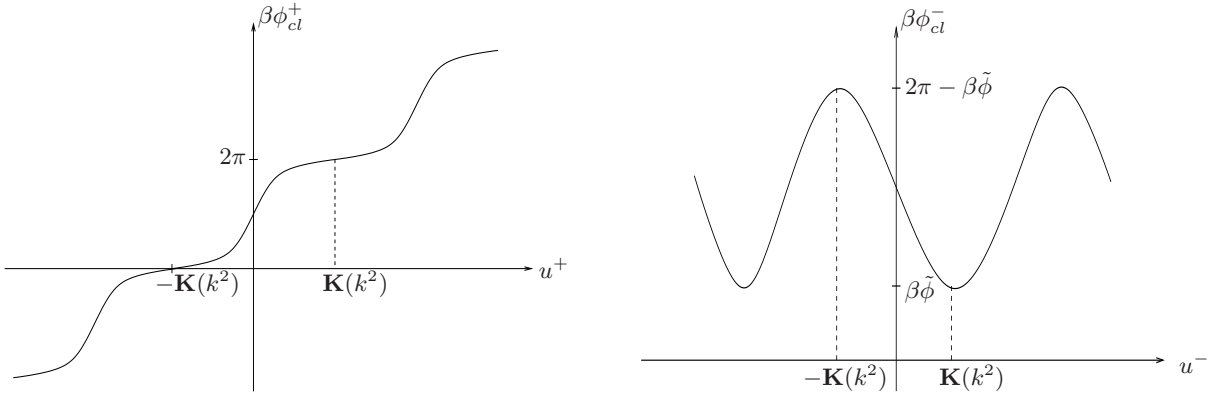


Figure 7: Solutions of eq. (4.1), $A > 0$ (left hand side), $-2 < A < 0$ (right hand side).

The solution (4.3) satisfies quasiperiodic b.c.

$$\phi(x+R) = \phi(x) + \frac{2\pi}{\beta} , \quad (4.5)$$

provided the circumference R of the cylinder is identified with $R = \frac{1}{m} 2 k \mathbf{K}(k^2)$. The complete semiclassical quantization of this background has been performed in [8]. It is worth to recall here the UV limit of the corresponding energy levels, given by

$$\begin{aligned} \frac{\mathcal{E}_{\{k_n\}}(R)}{m} &= \frac{2\pi}{r} \left(\frac{\pi}{\beta^2} + \sum_n k_n n \right) - \frac{1}{4} + \frac{1}{\beta^2} r - \frac{1}{8} \left(\frac{r}{2\pi} \right)^2 + \\ &- \left(\frac{r}{2\pi} \right)^3 \left[\frac{1}{8} \zeta(3) - \frac{1}{4} (2 \log 2 - 1) - \frac{\pi}{2\beta^2} + \sum_n k_n \frac{n}{4n^2 - 1} \right] + \dots \end{aligned} \quad (4.6)$$

where $\{k_n\}$ is a set of integers defining a particular excited state of the kink.

We will now present a similar analysis for the solution (4.4), which satisfies antiperiodic b.c.

$$\phi(x + R) = -\phi(x) + \frac{2\pi}{\beta}, \quad (4.7)$$

if it is defined on a cylinder of circumference

$$R = \frac{1}{m} 2\mathbf{K}(k^2). \quad (4.8)$$

Similarly to the kink (2.20) studied in the ϕ^4 case, the solution (4.4) tends to the standard infinite-volume soliton (4.2) for $A \rightarrow 0$, when R goes to infinity. In the other limit $A \rightarrow -2$, which corresponds to $mR \rightarrow \pi$, (4.4) goes to the constant solution

$$\phi_{cl}(x) \equiv \frac{\pi}{\beta}, \quad (4.9)$$

which identically satisfies the antiperiodic b.c. (4.7) and can be therefore used as the background in the interval $0 < mR < \pi$. Therefore, the classical energy associated to this kink background is

$$\mathcal{E}_{cl}(R) = \begin{cases} \frac{8m}{\beta^2} [\mathbf{E}(k) - \frac{1}{2}(1 - k^2)\mathbf{K}(k)] & \text{for } mR > \pi \\ \frac{2m}{\beta^2} mR & \text{for } mR < \pi \end{cases}, \quad (4.10)$$

and it is plotted in Fig. 8.

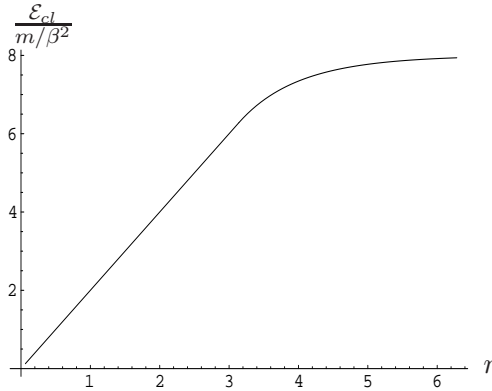


Figure 8: Classical energy (4.10)

The stability equation associated to (4.4) takes the form

$$\left\{ \frac{d^2}{d\bar{x}^2} + \bar{\omega}^2 + 1 - 2k^2 \operatorname{sn}^2 \bar{x} \right\} \eta(\bar{x}) = 0, \quad (4.11)$$

where

$$\bar{x} = mx, \quad \bar{\omega} = \frac{\omega}{m}. \quad (4.12)$$

This can be cast in the Lamé form with $N = 1$ (for the details, see Appendix B), which has the band structure shown in Fig. 9. Imposing then the antiperiodic boundary conditions (i.e.

selecting the values of $\bar{\omega}^2$ for which the Floquet exponent is an odd multiple of π), we obtain the simple eigenvalues $\bar{\omega}_0^2 = 0$ and

$$\bar{\omega}_1^2 = k, \quad (4.13)$$

and the infinite series of double eigenvalues

$$\bar{\omega}_n^2 \equiv \frac{2k^2 - 1}{3} - \mathcal{P}(iy_n) \quad (4.14)$$

in the band $\bar{\omega}^2 > k^2$, with y_n defined by

$$F = 2\mathbf{K} i \zeta(iy_n) + 2y_n \zeta(\mathbf{K}) = (2n - 1)\pi, \quad n = 2, 3, \dots \quad (4.15)$$

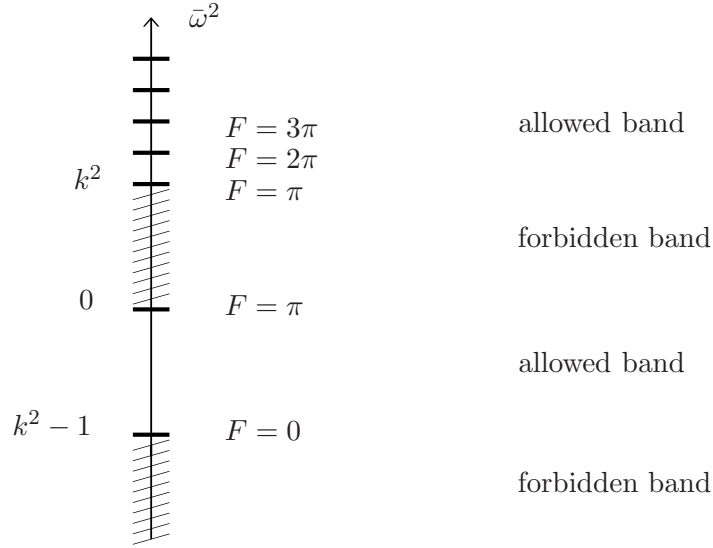


Figure 9: Spectrum of eq. (4.11)

It is easy to see that in the IR limit ($A \rightarrow 0$) this spectrum goes to the one related to the standard background (4.2). In order to complete the spectrum, also at values $mR < \pi$, we have to glue the frequencies (4.13) and (4.14) with the ones obtained by quantizing the constant solution (4.9). We therefore obtain

$$\bar{\omega}_1^2 = \begin{cases} k & \text{for } mR > \pi \\ -1 + \frac{\pi^2}{m^2 R^2} & \text{for } mR < \pi \end{cases}, \quad (4.16)$$

and

$$\bar{\omega}_n^2 = \begin{cases} \frac{2k^2 - 1}{3} - \mathcal{P}(iy_n) & \text{for } mR > \pi \\ -1 + (2n - 1)^2 \frac{\pi^2}{m^2 R^2} & \text{for } mR < \pi \end{cases}, \quad (4.17)$$

which are plotted in Fig. 10.

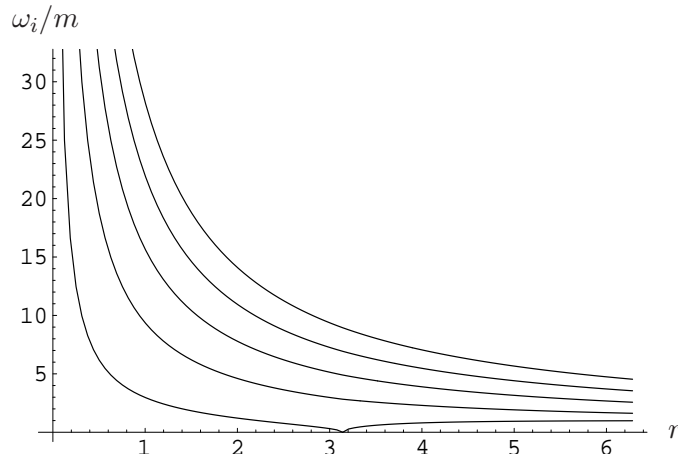


Figure 10: The first few levels defined in (4.16) and (4.17)

The study of the corresponding scaling functions can be performed along the same lines illustrated for the broken ϕ^4 theory. One easily obtains the UV limit of the ground state energy in the form

$$E_0(R) \simeq \frac{2\pi}{R} \left(h + \bar{h} - \frac{c}{12} \right) + \mathcal{B} R + \dots, \quad (4.18)$$

with $h + \bar{h} = 1/8$ and

$$\mathcal{B} = m^2 \left(\frac{2}{\beta^2} - \frac{\log 2}{2\pi} \right).$$

Therefore, we have seen explicitly how the two types of kink (4.3) and (4.4), although they have the same IR limit, display different energy levels in finite volume, and in particular different UV limits, describing both twisted and untwisted sectors of $c = 1$ CFT.

5 Open problems and discussion

In this paper we have applied the semiclassical method to derive analytic expressions for the energy levels of the broken ϕ^4 theory on a cylinder with antiperiodic b.c.. Although this analysis is technically similar to the one performed in [8] for the Sine–Gordon model in the one–kink sector, various conceptual differences have emerged.

The derivation of analytic expressions for the finite–volume semiclassical energy levels in the ϕ^4 model is based on two important ingredients: the explicit form of the kink solution (2.20) and the eigenvalues (2.31, 2.32) of the $N = 2$ Lamé equation. Therefore its extension to ϕ^6 and higher order $p \geq 5$ LG potentials (1.1) requires the knowledge of the corresponding finite–volume kinks as well as certain properties of the solutions of their stability equations (2.1). Consider a

family of symmetric (or “hyperelliptic”) LG potentials

$$\begin{aligned} V_p^{\text{odd}} &= \frac{1}{2} \prod_{k=1}^{\frac{p-1}{2}} (\phi^2 - a_k^2)^2 \quad \text{for } p = 3, 5, \dots \\ V_p^{\text{even}} &= \frac{1}{2} \phi^2 \prod_{k=1}^{\frac{p-2}{2}} (\phi^2 - a_k^2)^2 \quad \text{for } p = 4, 6, \dots \end{aligned} \quad (5.1)$$

Their static kink solutions, i.e. the solutions of the first order equation

$$\frac{1}{2} \left(\frac{d\phi_{\text{cl}}}{dx} \right)^2 = V_p(\phi_{\text{cl}}) + A = \frac{1}{2} \prod_{l=1}^{p-1} (\phi_{\text{cl}}^2 - b_l) ,$$

where $A = -V(\phi_0)$, $b_l = b_l(a_k)$ and $b_1 = \phi_0^2$, are given for both odd and even p by the inverse of the following hyperelliptic integrals:

$$\pm 2x = \int_{\phi_0^2}^{\phi_{\text{cl}}^2(x)} \frac{dz}{\sqrt{z \prod_{l=1}^{p-1} (z - b_l)}} . \quad (5.2)$$

In the case $p = 4$ (i.e. for the ϕ^6 model) the integral in (5.2) is of elliptic type and the corresponding finite-volume kink has the explicit form

$$\phi_{\text{cl}}^{(p=4)}(x) = \frac{\sqrt{b_1}}{\sqrt{1 - \left(1 - \frac{b_1}{b_2}\right) \text{sn}^2\left(\sqrt{b_2(b_3 - b_1)} x, k\right)}} , \quad (5.3)$$

where

$$\begin{aligned} k^2 &= \left(\frac{b_3}{b_2} \right) \frac{b_2 - b_1}{b_3 - b_1} , \\ \begin{cases} b_2 = \frac{1}{2} \left(2a_1^2 - b_1 - \sqrt{b_1(4a_1^2 - 3b_1)} \right) \\ b_3 = \frac{1}{2} \left(2a_1^2 - b_1 + \sqrt{b_1(4a_1^2 - 3b_1)} \right) \end{cases} . \end{aligned}$$

This background satisfies the boundary conditions

$$\phi_{\text{cl}}(R) = \sqrt{b_1} + \sqrt{b_2} - \phi_{\text{cl}}(0) ,$$

provided we identify the size of the system as

$$R = \frac{1}{\sqrt{b_2(b_3 - \phi_0^2)}} \mathbf{K} .$$

Although for $p > 4$ the kink solutions are not given in an explicit form, one can easily derive their stability equation through the change of variable $z = \phi_{\text{cl}}^2(x)$:

$$\frac{d^2\eta(z)}{dz^2} + \frac{1}{2} \left(\frac{1}{z} + \sum_{l=1}^{p-1} \frac{1}{z - b_l} \right) \frac{d\eta(z)}{dz} - \frac{V_p''(z) - \omega^2}{2z \prod_{l=1}^{p-1} (z - b_l)} \eta(z) = 0 , \quad (5.4)$$

with the antiperiodic b.c. expressed as

$$\eta(z(R)) = -\eta(z(0)) \quad , \quad (5.5)$$

where R is the smallest real period of the hyperelliptic integral (5.2). The above second order ODE's with $p+1$ regular singular points (at $z = 0, b_l, \infty$) represents a generalization [30] of the Lamé equation in the so-called algebraic form¹⁰

$$\frac{d^2\eta(z)}{dz^2} + \frac{1}{2} \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-a} \right) \frac{d\eta(z)}{dz} - \frac{N(N+1)z-\lambda}{2z(z-1)(z-a)} \eta(z) = 0 \quad , \quad (5.6)$$

which coincides with (5.4) for $N = 2, p = 3$ and $V_3''(z) = 6z - 2a_1^2$, i.e. for the ϕ^4 potential analyzed in Sect. 2.

Hence the derivation of the semiclassical scaling functions of the generic $p \geq 4$ LG models (5.1) defined on the cylinder reduces to the problem of construction of the solutions and eigenvalues of the generalized Lamé equation (5.4) for antiperiodic b.c. (5.5). For $p \geq 4$ this is an interesting open problem, whose analytical or numerical solutions will provide the necessary ingredients for calculations of the corresponding energy levels.

Finally, it is worth mentioning few more research directions that arise as natural developments of the analysis carried out here. One of them consists of the determination of the energy levels in the presence of different boundary conditions. Equally interesting is to extend our computations to higher loop orders: although the one-loop quantization around a kink background is more powerful than standard perturbative techniques, we have seen however that it is not yet accurate enough to identify the Ising critical point in the phase diagram of the ϕ^4 theory. The last point we would like to mention is the study of symmetry restoration in finite volume for antiperiodic boundary conditions. This phenomenon is well understood in the vacuum sector (i.e. for periodic b.c. [25, 3]) but it is still an open problem in the kink sector, and it may be fruitfully investigated within the semiclassical approach.

Acknowledgements

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A Elliptic integrals and Jacobi's elliptic functions

In this appendix we collect the definitions and basic properties of the elliptic integrals and functions used in the text. Exhaustive details can be found in [26].

¹⁰The same equation is expressed in the alternative Weierstrass form in (B.1).

The complete elliptic integrals of the first and second kind, respectively, are defined as

$$\mathbf{K}(k^2) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \quad \mathbf{E}(k^2) = \int_0^{\pi/2} d\alpha \sqrt{1 - k^2 \sin^2 \alpha}. \quad (\text{A.1})$$

The parameter k , called elliptic modulus, has to be bounded by $k^2 < 1$. It turns out that the elliptic integrals are nothing but specific hypergeometric functions, which can be easily expanded for small k :

$$\begin{aligned} \mathbf{K}(k^2) &= \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right) = \frac{\pi}{2} \left\{ 1 + \frac{1}{4} k^2 + \frac{9}{64} k^4 + \dots + \left[\frac{(2n-1)!!}{2^n n!} \right]^2 k^{2n} + \dots \right\}, \\ \mathbf{E}(k^2) &= \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1; k^2\right) = \frac{\pi}{2} \left\{ 1 - \frac{1}{4} k^2 - \frac{3}{64} k^4 + \dots - \left[\frac{(2n-1)!!}{2^n n!} \right]^2 \frac{k^{2n}}{2n-1} + \dots \right\}. \end{aligned}$$

Furthermore, for $k^2 \rightarrow 1$, they admit the following expansion in the so-called complementary modulus $k' = \sqrt{1 - k^2}$:

$$\begin{aligned} \mathbf{K}(k^2) &= \log \frac{4}{k'} + \left(\log \frac{4}{k'} - 1 \right) \frac{k'^2}{4} + \dots, \\ \mathbf{E}(k^2) &= 1 + \left(\log \frac{4}{k'} - \frac{1}{2} \right) \frac{k'^2}{2} + \dots. \end{aligned}$$

Note that the complementary elliptic integral of the first kind is defined as

$$\mathbf{K}'(k^2) = \mathbf{K}(k'^2).$$

The function $\text{am}(u, k^2)$, depending on the parameter k , and called Jacobi's elliptic amplitude, is defined through the first order differential equation

$$\left(\frac{d \text{am}(u)}{du} \right)^2 = 1 - k^2 \sin^2 [\text{am}(u)], \quad (\text{A.2})$$

and it is doubly quasi-periodic in the variable u :

$$\text{am}(u + 2n\mathbf{K} + 2im\mathbf{K}') = n\pi + \text{am}(u).$$

The Jacobi's elliptic function $\text{sn}(u, k^2)$, defined through the equation

$$\left(\frac{d \text{sn} u}{du} \right)^2 = (1 - \text{sn}^2 u) (1 - k^2 \text{sn}^2 u), \quad (\text{A.3})$$

is related to the amplitude by $\text{sn} u = \sin(\text{am} u)$, and it is doubly periodic:

$$\text{sn}(u + 4n\mathbf{K} + 2im\mathbf{K}') = \text{sn}(u).$$

B Lamé equation

The second order differential equation

$$\left\{ \frac{d^2}{du^2} - E - N(N+1)\mathcal{P}(u) \right\} f(u) = 0 , \quad (\text{B.1})$$

where E is a real quantity, N is a positive integer and $\mathcal{P}(u)$ denotes the Weierstrass function, is known under the name of N -th Lamé equation. The function $\mathcal{P}(u)$ is a doubly periodic solution of the first order equation (see [26])

$$\left(\frac{d\mathcal{P}}{du} \right)^2 = 4(\mathcal{P} - e_1)(\mathcal{P} - e_2)(\mathcal{P} - e_3) , \quad (\text{B.2})$$

whose characteristic roots e_1, e_2, e_3 uniquely determine the half-periods ω and ω' , defined by

$$\mathcal{P}(u + 2n\omega + 2m\omega') = \mathcal{P}(u) .$$

The stability equation (2.29), related to the broken ϕ^4 theory, can be identified with eq. (B.1) for $N = 2$, $u = \frac{\bar{\phi}_0}{\sqrt{2}} \bar{x} + i\mathbf{K}'$ and $E = (1 + k^2)(1 - \bar{\omega}^2)$; also the stability equation (4.11), encountered in the analysis of the Sine-Gordon model, can be identified with eq. (B.1), in this case with $N = 1$, $u = \bar{x} + i\mathbf{K}'$ and $E = \frac{2k^2-1}{3} - \bar{\omega}^2$. Both these identifications hold in virtue of the relation between $\mathcal{P}(u)$ and the Jacobi elliptic function $\text{sn}(u, k)$ (see formulas 8.151 and 8.169 of [26]):

$$k^2 \text{sn}^2(\bar{x}, k) = \mathcal{P}(\bar{x} + i\mathbf{K}') + \frac{k^2 + 1}{3} . \quad (\text{B.3})$$

Relation (B.3) is valid if the characteristic roots of $\mathcal{P}(u)$ are expressed in terms of k^2 as

$$e_1 = \frac{2 - k^2}{3} , \quad e_2 = \frac{2k^2 - 1}{3} , \quad e_3 = -\frac{1 + k^2}{3} , \quad (\text{B.4})$$

and, as a consequence, the real and imaginary half periods of $\mathcal{P}(u)$ are given by the elliptic integrals of the first kind

$$\omega = \mathbf{K}(k) , \quad \omega' = i\mathbf{K}'(k) . \quad (\text{B.5})$$

All the properties of Weierstrass functions that we will use in the following are specified to the case when this identification holds.

We will now present the solutions of the Lamé equation for $N = 1$ and $N = 2$, which have been derived in [27, 28] together with more complicated cases.

In the case $N = 1$ the two linearly independent solutions of (B.1) are given by

$$f_{\pm a}(u) = \frac{\sigma(u \pm a)}{\sigma(u)} e^{\mp u \zeta(a)} , \quad (\text{B.6})$$

where a is an auxiliary parameter defined through $\mathcal{P}(a) = E$, and $\sigma(u)$ and $\zeta(u)$ are other kinds of Weierstrass functions:

$$\frac{d\zeta(u)}{du} = -\mathcal{P}(u) , \quad \frac{d \log \sigma(u)}{du} = \zeta(u) , \quad (\text{B.7})$$

with the properties

$$\begin{aligned}\zeta(u + 2\mathbf{K}) &= \zeta(u) + 2\zeta(\mathbf{K}) , \\ \sigma(u + 2\mathbf{K}) &= -e^{2(u+\mathbf{K})\zeta(\mathbf{K})}\sigma(u) .\end{aligned}\tag{B.8}$$

As a consequence of eq. (B.8) one obtains the Floquet exponent of $f_{\pm a}(u)$, defined as

$$f(u + 2\mathbf{K}) = f(u)e^{iF(a)} ,\tag{B.9}$$

in the form

$$F(\pm a) = \pm 2i [\mathbf{K} \zeta(a) - a \zeta(\mathbf{K})] .\tag{B.10}$$

The spectrum in the variable E of eq. (B.1) with $N = 1$ is divided in allowed/forbidden bands depending on whether $F(a)$ is real or complex for the corresponding values of a . We have that $E < e_3$ and $e_2 < E < e_1$ correspond to allowed bands, while $e_3 < E < e_2$ and $E > e_1$ are forbidden bands. Note that if we exploit the periodicity of $\mathcal{P}(a)$ and redefine $a \rightarrow a' = a + 2n\omega + 2m\omega'$, this only shifts F to $F' = F + 2m\pi$.

The solutions of the Lamé equation with $N = 2$ are given by

$$f(u) = \frac{\sigma(u+a)\sigma(u+b)}{\sigma^2(u)} e^{-u[\zeta(a)+\zeta(b)]} ,\tag{B.11}$$

where a and b are two auxiliary parameters satisfying the constraints

$$\begin{cases} 3\mathcal{P}(a) + 3\mathcal{P}(b) = E \\ \mathcal{P}'(a) + \mathcal{P}'(b) = 0 \end{cases} ,\tag{B.12}$$

and $\sigma(u)$ and $\zeta(u)$ are defined in (B.7). The Floquet exponent of $f(u)$ is now given by

$$F = 2i \{ \mathbf{K} [\zeta(a) + \zeta(b)] - (a+b)\zeta(\mathbf{K}) \} .\tag{B.13}$$

The spectrum in the variable E of eq. (B.1) with $N = 2$ is divided in allowed (A) and forbidden (F) bands depending on whether F is real or complex for the corresponding values of a and b , as shown in Fig. 11.

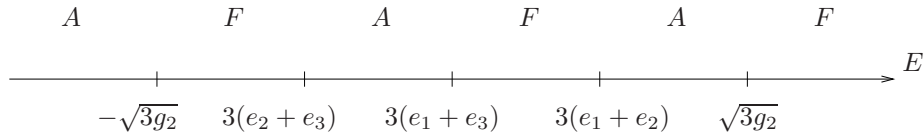


Figure 11: Spectrum of eq.(B.1) with $N = 2$, where e_1, e_2, e_3 are the roots of \mathcal{P} and $g_2 = 2(e_1^2 + e_2^2 + e_3^2)$.

Finally, it is worth mentioning that the function $\zeta(u)$ admits a series representation [29] that is very useful for our purposes in the text:

$$\zeta(u) = \frac{\pi}{2\mathbf{K}} \cot\left(\frac{\pi u}{2\mathbf{K}}\right) + \left(\frac{\mathbf{E}}{\mathbf{K}} + \frac{k^2 - 2}{3}\right) u + \frac{2\pi}{\mathbf{K}} \sum_{n=1}^{\infty} \frac{h^{2n}}{1 - h^{2n}} \sin\left(\frac{n\pi u}{\mathbf{K}}\right) ,\tag{B.14}$$

where $h = e^{-\pi \mathbf{K}'/\mathbf{K}}$. The small- k expansion of this expression gives

$$\begin{aligned} \zeta(u) = & \left(\cot u + \frac{u}{3} \right) + \frac{k^2}{12} (u - 3 \cot u + 3u \cot^2 u) + \\ & + \frac{k^4}{64} (-3u + (4u^2 - 5) \cot u + u \cot^2 u + 4u^2 \cot^3 u + \sin 2u) + \dots \end{aligned} \quad (\text{B.15})$$

(note that $h \approx \left(\frac{k}{4}\right)^2 + O(k^4)$). A similar expression takes place for $\mathcal{P}(u)$, by noting that $\mathcal{P}(u) = -\frac{d\zeta(u)}{du}$.

C Ground state energy regularization at $r \approx \pi$

We present in this appendix the evaluation of the ground state energy (2.36) for $r \lesssim \pi$ and $r \gtrsim \pi$, comparing the two corresponding expressions at the point $r = \pi$.

In the case $r \lesssim \pi$, we obtain

$$\frac{E_0}{m}(r) = A_- + \sqrt{2} \sqrt{1 - \frac{r}{\pi}} + B_- \left(1 - \frac{r}{\pi}\right) + \dots, \quad (\text{C.1})$$

where the coefficients A_- and B_- are defined as

$$\begin{aligned} A_- &= \frac{m^2}{\lambda} \frac{\pi}{4} + \sum_{n=1}^{\infty} \sqrt{(2n-1)^2 - 1} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{(2n)^2 + 2}} - \sum_{n=1}^{\infty} \sqrt{(2n)^2 + 2} - \frac{1}{4\sqrt{2}}, \\ B_- &= -\frac{m^2}{\lambda} \frac{\pi}{4} + \sum_{n=2}^{\infty} \frac{(2n-1)^2}{\sqrt{(2n-1)^2 - 1}} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{(2n)^2}{[(2n)^2 + 2]^{3/2}} - \sum_{n=1}^{\infty} \frac{(2n)^2}{\sqrt{(2n)^2 + 2}}. \end{aligned}$$

Expanding in $\frac{1}{(2n-1)}$ and $\frac{1}{(2n)}$, we obtain

$$\begin{aligned} A_- &= \frac{m^2}{\lambda} \frac{\pi}{4} + \sum_{n=1}^{\infty} (2n-1) - \sum_{n=1}^{\infty} (2n) - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(2n)} - \frac{1}{4\sqrt{2}} - C_-, \\ B_- &= -\frac{m^2}{\lambda} \frac{\pi}{4} + \sum_{n=1}^{\infty} (2n-1) - \sum_{n=1}^{\infty} (2n) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(2n)} - 1 + D_-, \end{aligned} \quad (\text{C.2})$$

where C_- and D_- are finite constants given by

$$\begin{aligned} C_- &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!!}{(k+1)! 2^{2k+1}} \left[\frac{\zeta(2k+1, 1/2)}{2^{k+1}} - \frac{3k+1}{2} \zeta(2k+1) \right], \\ D_- &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2k-1)!!}{k! 2^{2k+1}} \left[\frac{2k+3}{k+1} \frac{\zeta(2k+1, 1/2)}{2^{k+1}} + \frac{(3k+1)(2k+1)}{2(k+1)} \zeta(2k+1) - 2^{k+1} \right], \end{aligned}$$

with numerical values $C_- \simeq 0.018$, $D_- \simeq 0.39$, and the functions in this expressions are defined as

$$\begin{cases} n! = 1 \cdot 2 \cdot \dots \cdot n \\ (2n+1)!! = 1 \cdot 3 \cdot \dots \cdot (2n+1) \end{cases}, \quad \begin{cases} \zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} \\ \zeta(p, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^p} \end{cases}.$$

The individually divergent series present in (C.2) combine to give a finite result, in virtue of the relations

$$\begin{aligned} \sum_{n=0}^{\infty} (2n+1) - \sum_{n=1}^{\infty} (2n) &= 2 [\zeta(-1, 1/2) - \zeta(-1)] = 2 \left[\frac{1}{24} + \frac{1}{12} \right] , \\ \sum_{n=0}^{\infty} \frac{1}{(2n+1)} - \sum_{n=1}^{\infty} \frac{1}{(2n)} &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2 . \end{aligned}$$

Therefore, the final expressions for the coefficients A_- and B_- are

$$\begin{aligned} A_- &= \frac{m^2}{\lambda} \frac{\pi}{4} + \frac{1}{4} - \frac{1}{2} \log 2 - \frac{1}{4\sqrt{2}} - C_- , \\ B_- &= -\frac{m^2}{\lambda} \frac{\pi}{4} + \frac{1}{4} + \frac{1}{2} \log 2 - 1 + D_- . \end{aligned}$$

The other case $mR \gtrsim \pi$ can be similarly treated, being more complicated only from the technical point of view. In fact, it requires to compare, in the limit $k \rightarrow 0$, the behavior of classical energy and stability frequencies, defined in (2.24), (2.31) and (2.32) respectively, with the one of the scaling variable, defined in (2.21). The expansions of elliptic integrals and Weierstrass functions, necessary for this purpose, can be found in Appendices A and B. Since the scaling variable has the small- k behaviour

$$r = \pi \left[1 + \frac{3}{4} k^2 + \dots \right] , \quad (\text{C.3})$$

it is easy to see that

$$\frac{\mathcal{E}_{\text{cl}}}{m} = \frac{m^2}{\lambda} \frac{\pi}{4} \left(1 + \frac{3}{4} k^2 \right) + \dots = \frac{m^2}{\lambda} \frac{\pi}{4} + \frac{m^2}{4\lambda} (r - \pi) + \dots \quad (\text{C.4})$$

and

$$\frac{\omega_1}{m} = \sqrt{3} k + \dots = 2 \sqrt{\frac{r}{\pi} - 1} + \dots . \quad (\text{C.5})$$

The frequencies (2.32) have the most implicit expression in term of r . Noting that in the highest band $\bar{\omega}^2 > 1 + \frac{2\sqrt{k^4 - k^2 + 1}}{1 + k^2}$ the auxiliary parameters a and b are related as $a = -b^*$, we can conveniently parameterize a_n and b_n in (2.33) as

$$\begin{cases} a_n = -x_n + i y_n \\ b_n = x_n + i y_n \end{cases} \quad (\text{C.6})$$

Expanding equations (2.33) for small k , we obtain

$$\begin{cases} x_n = \frac{1}{2} \arcsin \left(\sqrt{\frac{3}{(2n+1)^2 - 1}} \right) \left[1 + \frac{k^2}{4} + \dots \right] \\ y_n = \frac{1}{2} \operatorname{arcsinh} \left(3 \sqrt{\frac{(2n+1)^2}{[(2n+1)^2 - 1][(2n+1)^2 - 4]}} \right) \left[1 + \frac{k^2}{4} + \dots \right] \end{cases} \quad (\text{C.7})$$

and therefore

$$\bar{\omega}_n^2 = [(2n+1)^2 - 1] \left\{ 1 - \frac{3}{2} k^2 \frac{(2n+1)^2 - 2}{(2n+1)^2 - 1} + \dots \right\} . \quad (\text{C.8})$$

Comparing this with (C.3) we finally obtain

$$\frac{\omega_n}{m}(r) = \sqrt{(2n+1)^2 - 1} - \frac{(2n+1)^2 - 2}{\sqrt{(2n+1)^2 - 1}} \left(\frac{r}{\pi} - 1 \right) + \dots . \quad (\text{C.9})$$

Therefore, the ground state energy has the behaviour

$$\frac{E_0}{m}(r) = A_+ + \sqrt{\frac{r}{\pi} - 1} + B_+ \left(\frac{r}{\pi} - 1 \right) + \dots , \quad (\text{C.10})$$

where $A_+ = A_-$ and $B_+ = B_-$.

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